Improved Multilevel Hadamard Matrices and their Generalizations over the Gaussian and Hamiltonian Integers

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Abstract

Multilevel Hadamard matrices (MHMs), whose entries are integers as opposed to the traditional restriction to $\{\pm 1\}$, have been introduced as a way to construct multilevel zero-correlation zone sequences for use in approximately synchronized code division multiple access (AS-CDMA) systems. This paper provides a construction technique to produce $2^m \times 2^m$ MHMs whose 2^m alphabet entries form an arithmetic progression, up to sign. This construction improves upon existing constructions because it permits control over the spacing and overall span of the MHM entries. MHMs with such regular alphabets are a more direct generalization of traditional Hadamard matrices and are thus expected to be more useful in applications analogous to those of Hadamard matrices. This paper also introduces mixed-circulant MHMs which provide a certain advantage over known circulant MHMs of the same size.

MHMs over the Gaussian (complex) and Hamiltonian (quaternion) integers are introduced. Several constructions are provided, including a generalization of the arithmetic progression construction for MHMs over real integers. Other constructions utilize amicable pairs of MHMs and c-MHMs, which are introduced as natural generalizations of amicable orthogonal designs and c-Hadamard matrices, respectively. The constructions are evaluated against proposed criteria for interesting and useful MHMs over these generalized alphabets.

1 Overview

This paper presents improved multilevel Hadamard matrices (MHMs) and introduces MHMs over the Gaussian (complex) and Hamiltonian (quaternion) integers. Background information on MHMs is provided in Subsection 2.1 and a recursive algorithm for order 2^m MHMs over arithmetic progression alphabets is provided in Subsection 2.2. These MHMs represent an improvement over existing MHMs due to the regular spacing and the controlled span of their alphabet entries. In Subsection 2.3, we build order $2m$ mixed-circulant MHMs using pairs of order m MHMs. We also generalize this construction to use pairs of c-MHMs, defined herein as a natural generalization of c-Hadamard matrices [10].

$\mathbb{C}$-MHMs over the Gaussian integers are introduced in Section 3. Several constructions of $\mathbb{C}$-MHMs are proposed, including a construction using amicable pairs of MHMs, a construction using amicable pairs of c-MHMs, and a generalization of the arithmetic progression construction for MHMs over the real integers. We introduce $\mathbb{Q}$-MHMs over the Hamiltonian integers in Section 4 and provide constructions that are generalizations of those proposed for $\mathbb{C}$-MHMs. Conclusions and open problems are reviewed in Section 5.

2 Improved Multilevel Hadamard Matrices

2.1 Introduction to MHMs

In 2006, Trinh *et al.* [17] introduced multilevel Hadamard matrices for use in approximately synchronized code division multiple access (AS-CDMA) systems [4, 17]. Here, we provide a clarified definition as follows:

Definition 1. *A* multilevel Hadamard matrix *(MHM) A of order n is an $n \times n$ matrix such that each row and column contains some permutation of the integer alphabet $\mathcal{A} = \{a_0, a_1, ..., a_{n-1}\}$, up to sign and where the a_i can be nondistinct, such that $AA^T = A^TA = \sum_{i=0}^{n-1} |a_i|^2 I_n$, where I_n is the order n identity matrix.*

Previously it had been understood but not stated that each row and each column must contain permutations of the same entries, up to sign. If the entries a_i are restricted to $\{\pm 1\}$, then we obtain traditional Hadamard matrices. On the other hand, if the a_i are generalized to real variables, then we obtain real orthogonal designs. Thus, MHMs may be viewed as an intermediary between Hadamard matrices, which have been applied in a variety of areas, including CDMA spreading systems, error control coding, optical multiplexing, and the design of statistical experiments [3, 15], and real orthogonal designs, which have also proven useful for wireless communications [11, 18].

We introduce the *entry count* of an MHM as follows:

Definition 2. *The* entry count *of an MHM A is the number of distinct entries up to sign in A; in other words, the number of alphabet entries with distinct absolute values.*

In general, we aim to achieve an entry count that is as high as possible. An order n MHM with entry count n is considered the most interesting since it provides more variety in element choice and more closely resembles a real orthogonal design of full rate [11]. This notion had been captured previously through the introduction of the *rate* of an MHM, defined as the the number of elements of distinct absolute value divided by the order of the matrix [1]. Given an order n MHM, it has entry count n if and only if it has rate 1. A constructive proof has previously shown that there exist entry count n MHMs of any order n [1]; accordingly, our work here focuses primarily on order n MHMs of entry count n. We describe such MHMs as *full entry count* MHMs.

2.2 MHMs Over Arithmetic Progression Alphabets

The existing construction [1] of full entry count order n MHMs has a drawback: the alphabets of the constructed MHMs involve entries of a geometric progression, so the entries are not evenly spaced and they grow exponentially large as the order of the MHM increases. These properties are likely undesirable for applications, which would more naturally require entries that are evenly spaced within a limited range, more directly generalizing traditional Hadamard matrices over $\{\pm 1\}$. In this subsection, we present a recursive construction for MHMs whose entries form an arithmetic progression, up to sign, where the initial integer and common difference can be chosen arbitrarily. These arithmetic progression MHMs achieve full entry count, however they are constructed only for orders $n = 2^m$. This construction was motivated by Trinh *et al.*'s example of an order $2^m = 4$ MHM whose entries are consecutive integers beginning with 1 [17]. We expect the additional control we provide by generalizing to arithmetic progression alphabets (up to sign) will be useful. As shown below, our construction also implies the existence of order 2^m MHMs whose entries are all primes, bringing us to a stronger intermediary between Hadamard matrices and real orthogonal designs.

We begin with some definitions. Recall first that a matrix M is *symmetric* if $M = M^T$, is *skew-symmetric* if $M = -M^T$, and is *skew* if $M + M^T = zI$ where z is some constant.

Definition 3. *Given a matrix $X = [x_{ij}]$, the* sign matrix $S = [s_{ij}]$ *corresponding to X is defined as a matrix of the same size where $s_{ij} = x_{ij}/|x_{ij}|$. In other words, S is a matrix over $\{\pm 1\}$ that encodes the signs of the corresponding entries of X.*

Definition 4. *The* half-identity matrix *of order 2^m is defined as $J_m = \begin{bmatrix} I_{2^{m-1}} & 0 \\ 0 & -I_{2^{m-1}} \end{bmatrix}$, where $I_{2^{m-1}}$ represents the identity matrix of order 2^{m-1}.*

Note that $J_m = J_m^T$ and $J_m^2 = I$. I is used here and will be used throughout to represent the identity matrix of the most appropriate size in context. Note also that although half-identity matrices exist for all even orders, we only consider those with orders 2^m.

Algorithm 1. *Given two positive integers a and d, let $X_1 = \begin{bmatrix} a & a+d \\ a+d & -a \end{bmatrix}$, and for $k > 1$, let $X_{k+1} = \begin{bmatrix} X_k & X_k + d2^k S_k \\ X_k J_k + d2^k S_k J_k & -X_k J_k \end{bmatrix}$, where S_k is the sign matrix corresponding to the X_k matrix and J_k is the half-identity matrix of order 2^k.*

Example 1. *For example, if $a = 3, d = 2$, then*

$$X_1 = \left[\begin{array}{cc} 3 & 5 \\ 5 & -3 \end{array}\right],$$

$$X_2 = \left[\begin{array}{cc|cc} 3 & 5 & 7 & 9 \\ 5 & -3 & 9 & -7 \\ \hline 7 & -9 & -3 & 5 \\ 9 & 7 & -5 & -3 \end{array}\right], \textit{ and}$$

$$X_3 = \left[\begin{array}{cccc|cccc} 3 & 5 & 7 & 9 & 11 & 13 & 15 & 17 \\ 5 & -3 & 9 & -7 & 13 & -11 & 17 & -15 \\ 7 & -9 & -3 & 5 & 15 & -17 & -11 & 13 \\ 9 & 7 & -5 & -3 & 17 & 15 & -13 & -11 \\ \hline 11 & 13 & -15 & -17 & -3 & -5 & 7 & 9 \\ 13 & -11 & -17 & 15 & -5 & 3 & 9 & -7 \\ 15 & -17 & 11 & -13 & -7 & 9 & -3 & 5 \\ 17 & 15 & 13 & 11 & -9 & -7 & -5 & -3 \end{array}\right].$$

We will now show that the matrices produced by Algorithm 1 are MHMs over arithmetic progression alphabets (up to sign) with initial term a and common difference d. We begin with a lemma; Appendix A contains straight-forward proofs by induction for each part.

Lemma 1. *Let X_m be an order 2^m matrix constructed via Algorithm 1, S_m be the corresponding sign matrix, and J_m be the half-identity matrix of order 2^m. Then,*

1. *S_m is a traditional Hadamard matrix;*
2. *$S_m X_m^T$ is skew; and*
3. *$X_m J_m S_m^T$ is symmetric.*

We are now prepared for our main result of this section:

Theorem 1. *For each $m \geq 1$, Algorithm 1 produces full entry count order 2^m MHMs over arithmetic progression alphabets.*

Proof. Consider the matrix $X_1 = \left[\begin{array}{cc} a & a+d \\ a+d & -a \end{array}\right]$ generated via Algorithm 1. Then,

$$X_1 X_1^T = X_1^T X_1 = \left[\begin{array}{cc} 2a^2 + 2ad + d^2 & 0 \\ 0 & 2a^2 + 2ad + d^2 \end{array}\right],$$

and it is clear that X_1 is an MHM of order 2^1 over the arithmetic progression alphabet $\{a, a+d\}$. Since X_1 was created via Algorithm 1, it follows respectively from the three parts of Lemma 1 that its sign matrix S_1 is a traditional Hadamard matrix, that $S_1 X_1^T$ is skew, and that $X_1 J_1 S_1^T$ is symmetric. We also confirm these conditions directly by noting that $S_1 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ is clearly a Hadamard matrix;

$$S_1 X_1^T = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} a & a+d \\ a+d & -a \end{bmatrix} = \begin{bmatrix} 2a+d & d \\ -d & 2a+d \end{bmatrix}$$

is clearly skew; and

$$X_1 J_1 S_1^T = \begin{bmatrix} a & a+d \\ a+d & -a \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} -d & 2a+d \\ 2a+d & d \end{bmatrix}$$

is clearly symmetric.

Let us assume that for some $k \geq 1$, X_k is a MHM of order 2^k that was created via Algorithm 1 and that has an arithmetic progression alphabet starting at a with common difference d. We will show this implies X_{k+1} (defined via Algorithm 1) is an MHM of order 2^{k+1} with an arithmetic progression alphabet starting at a with common difference d.

Taking the transpose of X_{k+1} and recalling that $J_k^T = J_k$ gives

$$X_{k+1}^T = \begin{bmatrix} X_k^T & d2^k J_k S_k^T + J_k X_k^T \\ X_k^T + d2^k S_k^T & -J_k X_k^T \end{bmatrix},$$

and then

$$X_{k+1} X_{k+1}^T = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

where

$$A = 2X_k X_k^T + d2^k X_k S_k^T + d2^k S_k X_k^T + d^2 2^{2k} S_k S_k^T;$$
$$B = d2^k X_k J_k S_k^T - d2^k S_k J_k X_k^T;$$
$$C = d2^k S_k J_k X_k^T - d2^k X_k J_k S_k^T; \text{ and}$$
$$D = d^2 2^{2k} S_k J_k J_k S_k^T + d2^k S_k J_k J_k X_k^T + d2^k X_k J_k J_k S_k^T + 2X_k J_k J_k X_k^T.$$

To show that X_{k+1} is a MHM, we must show that $A = D = bI_{2^k}$ for some scalar b, and $B = C = 0$. From our expression for A we get:

$$\begin{aligned} A &= 2wI_{2^k} + d^2 2^{2k} S_k S_k^T + d2^k (X_k S_k^T + S_k X_k^T) && (1) \\ &= 2wI_{2^k} + d^2 2^{2k} 2^k I_{2^k} + d2^k (X_k S_k^T + S_k X_k^T) && (2) \\ &= 2wI_{2^k} + d^2 2^{3k} I_{2^k} + d2^k z I_{2^k} && (3) \\ &= bI_{2^k} && (4) \end{aligned}$$

where (1) follows as X_k is an MHM by our inductive hypothesis and thus $X_k X_k^T = wI_{2^k}$ for some scalar w; (2) follows as S_k is an order 2^k Hadamard matrix by part 1 of Lemma 1 and thus $S_k S_k^T = 2^k I_{2^k}$; (3) follows as $S_k X_k^T$ is skew by part 2 of Lemma 1 and thus $X_k S_k^T + S_k X_k^T = zI_{2^k}$, where z is some integer. Therefore, (4) follows where b is some integer, namely $b = 2w + d^2 2^{3k} + d2^k z$. It follows similarly and using $J_m^2 = I$ that D is the same scalar multiple of I_{2^k}.

Now, since $J_k^T = J_k$, we can rewrite the expression for B as $B = d2^k((S_k J_k X_k^T)^T - S_k J_k X_k^T)$. Then, since $S_k J_k X_k^T$ is symmetric by part 3 of Lemma 1, we see that $B = 0$. It follows similarly that $C = 0$.

Thus, $X_{k+1} X_{k+1}^T = bI$. It can be shown similarly that $X_{k+1}^T X_{k+1} = bI$, using a similar lemma and similar proofs. It follows that X_{k+1} is an MHM.

Furthermore, since X_k has every row and column being a permutation of the arithmetic progression alphabet $\pm a, \pm(a+d), \ldots, \pm(a + (2^k - 1)d)$, the same holds for the rows and columns of $-(X_k)J_k$. It also follows that every row and column of the matrices $X_k + (d \cdot 2^k)S_k$ and $(X_k + (d \cdot 2^k)S_k)J_k$ are permutations of $\pm(a + 2^k d), \pm(a + (2^k + 1)d), \ldots, \pm(a + (2^{k+1} - 1)d)$. Therefore every row and column of X_{k+1} is a permutation of $\pm a, \pm(a + d), \ldots, \pm(a + (2^{k+1} - 1)d)$.

Thus, by the principle of mathematical induction, Algorithm 1 generates MHMs whose alphabets are in arithmetic progression for every order 2^m. Since a and d are positive, these MHMs are clearly of full entry count. $\square$

Notice that our algorithm and proof imply that we actually have more control over the spacing of entries than in a strict arithmetic progression. To illustrate, consider the following modification which can be substituted for X_{k+1} in Algorithm 1 and proved similarly as above:

$$X_{k+1} = \begin{bmatrix} X_k & X_k + d_k S_k \\ (X_k + d_k S_k)J_k & -(X_k)J_k \end{bmatrix}.$$

If we define $d_i = 2^i d$ for all $i = 1, 2, \ldots, k$ we get our original arithmetic progression construction. However, we can choose any set of integers to be $d_1, d_2, \ldots, d_k$ which would give additional control over the spacing of the entries in the resulting MHM. (We focused on arithmetic progressions, as we expect the regularity of such alphabets to be useful in future applications.) We provide the following example, P, which is an MHM constructed using the generalized algorithm with a =7, $d_1 = 6$, $d_2 = 16$, $d_3 = 30$, $d_4 = 534$.

$$
P = \begin{bmatrix}
7 & 13 & 23 & 29 & 37 & 43 & 53 & 59 & 541 & 547 & 557 & 563 & 571 & 577 & 587 & 593 \\
13 & -7 & 29 & -23 & 43 & -37 & 59 & -53 & 547 & -541 & 563 & -557 & 577 & -571 & 593 & -587 \\
23 & -29 & -7 & 13 & 53 & -59 & -37 & 43 & 557 & -563 & -541 & 543 & 587 & -593 & -571 & 577 \\
29 & 23 & -13 & -7 & 59 & 53 & -43 & -37 & 563 & 557 & -543 & -541 & 593 & 587 & -577 & -571 \\
37 & 43 & -53 & -59 & -7 & -13 & 23 & 29 & 571 & 577 & -587 & -593 & -541 & -547 & 557 & 563 \\
43 & -37 & -59 & 53 & -13 & 7 & 29 & -23 & 577 & -571 & -593 & 587 & -547 & 541 & 563 & -557 \\
53 & -59 & 37 & -43 & -23 & 29 & -7 & 13 & 587 & -593 & 571 & -577 & -557 & 563 & -541 & 547 \\
59 & 53 & 43 & 37 & -29 & -23 & -13 & -7 & 593 & 587 & 577 & 571 & -563 & -557 & -547 & -541 \\
541 & 547 & 557 & 563 & -571 & -577 & -587 & -593 & -7 & -13 & -23 & -29 & 37 & 43 & 53 & 59 \\
547 & -541 & 563 & -557 & -577 & 571 & -593 & 587 & -13 & 7 & -29 & 23 & 43 & -37 & 59 & -53 \\
557 & -563 & -541 & 543 & -587 & 593 & 571 & -577 & -23 & 29 & 7 & -13 & 53 & -59 & -37 & 43 \\
563 & 557 & -543 & -541 & -593 & -587 & 577 & 571 & -29 & -23 & 13 & 7 & 59 & 53 & -43 & -37 \\
571 & 577 & -587 & -593 & 541 & 547 & -557 & -563 & -37 & -43 & 53 & 59 & -7 & -13 & 23 & 29 \\
577 & -571 & -593 & 587 & 547 & -541 & -563 & 557 & -43 & 37 & 59 & -53 & -13 & 7 & 29 & -23 \\
587 & -593 & 571 & -577 & 557 & -563 & 541 & -547 & -53 & 59 & -37 & 43 & -23 & 29 & -7 & 13 \\
593 & 587 & 577 & 571 & 563 & 557 & 547 & 541 & -59 & -53 & -43 & -37 & -29 & -23 & -13 & -7
\end{bmatrix}
$$

This example is interesting because it contains all prime numbers. We chose to include an order 16 MHM with all prime numbers as it is the smallest case obtainable via our generalized algorithm that cannot be obtained by evaluating the entries of a full rate real orthogonal design (well-known to exist only for orders 1, 2, 4 and 8) at prime numbers. With a different rationale involving the original Algorithm 1, we have the following related corollary:

Corollary 1. *For any m, there exists a full entry count order 2^m MHM whose entries are all primes.*

Corollary 1 follows from Green and Tao's 2004 work proving the existence of arbitrarily long arithmetic progressions of primes [5]. Their proof is nonconstructive, but with it, we establish the existence of MHMs of all orders 2^m such that the entries are prime numbers in arithmetic progression. More concretely, at the time of writing, it is possible to use myriad databases available online to construct MHMs using up to 26 primes in arithmetic progression. In a sense, MHMs with prime entries are a stronger intermediary towards real orthogonal designs, and they may be more useful for certain applications as the entries share no common factors.

2.3 Mixed-Circulant MHMs

In this section, we introduce *mixed-circulant MHMs* whose entries follow a pattern that is sometimes seen in Latin squares and that bears resemblance to the pattern seen in circulant matrices.

Consider a matrix of even order n with columns $\mathbf{v}_1, \ldots, \mathbf{v}_n$ such that the odd columns $\mathbf{v}_1, \mathbf{v}_3, \ldots, \mathbf{v}_{n-1}$ contain the entries $c_0, c_1, \ldots, c_{n-1}$, beginning in this order in column $\mathbf{v}_1$ and then circulating down two steps per odd column and such that even columns $\mathbf{v}_2, \mathbf{v}_4, \ldots, \mathbf{v}_n$ contain entries $c_1, -c_0$, c_{n-1}, $-c_{n-2}$, $\ldots$, $(-1)^{k+1}c_k, \ldots, -c_2$, beginning in this order in column $\mathbf{v}_2$ and then circulating down two steps per even column. For example, consider the following 6×6 example C:

$$C = \begin{bmatrix} c_0 & c_1 & c_4 & c_3 & c_2 & c_5 \\ c_1 & -c_0 & c_5 & -c_2 & c_3 & -c_4 \\ c_2 & c_5 & c_0 & c_1 & c_4 & c_3 \\ c_3 & -c_4 & c_1 & -c_0 & c_5 & -c_2 \\ c_4 & c_3 & c_2 & c_5 & c_0 & c_1 \\ c_5 & -c_2 & c_3 & -c_4 & c_1 & -c_0 \end{bmatrix}$$

More formally, we have the following definition:

Definition 5. *A* mixed-circulant matrix $C = \{d_{ij}\}$ *of even order* n *is a square matrix with alphabet* $\{c_0, c_1, \ldots, c_{n-1}\}$ *such that*

$$d_{ij} = \begin{cases} c_{(i-j) \pmod n} & \text{if } j \text{ is odd} \\ c_{(-i+j) \pmod n} \cdot (-1)^{i+1} & \text{if } j \text{ is even} \end{cases}$$

Theorem 2. *Given a mixed-circulant matrix* A *with columns* $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$, $\mathbf{v}_x \cdot \mathbf{v}_y = 0$ *if* $x + y$ *is odd.*

Proof. Since exactly one among x and y is odd when $x+y$ is odd, without loss of generality let x be odd and y be even. Then,

$$\begin{aligned} \mathbf{v}_x \cdot \mathbf{v}_y &= \sum_{i=1}^{n} c_{i-x} c_{-i+y} . (-1)^{i+1} && (5) \\ &= \sum_{i_1=1}^{n} c_{i_1} c_{((y-x)-i_1)} . (-1)^{i_1+x+1} && (6) \\ &= \sum_{i_2=1}^{n} c_{((y-x)-i_2)} c_{i_2} . (-1)^{y-i_2+1} && (7) \\ &= \frac{1}{2} \sum_{i=1}^{n} c_i c_{((y-x)-i)} . \left((-1)^{i+x+1} + (-1)^{y-i+1}\right) && (8) \\ &= 0 && (9) \end{aligned}$$

where all subscripts are modulo n. Eqn. (6) follows from (5) by the substitution $i_1 = i - x$. Eqn. (7) follows from (5) by the substitution $i_2 = -i + y$. Eqn. (8) follows by taking the average of Equations (6) and (7). Then, as $(i + x + 1) + (y - i + 1) = x + y + 2$ is odd by our assumption, exactly one of the exponents on (-1) in Eqn. (8) is odd, hence Eqn. (9) follows. □

Thus, half of the columns in a mixed-circulant matrix are pairwise orthogonal regardless of alphabet. So, while a general order n MHM is required to satisfy $\binom{n}{2}$ orthogonality constraints, a mixed-circulant MHM must only satisfy $\lfloor \frac{n}{4} \rfloor$ orthogonality constraints, namely

$$\sum_{i=0}^{n-1} c_i c_{(i+2j) \pmod{n}} = 0, j = 1, 2 ... \left\lfloor \frac{n}{4} \right\rfloor.$$

This is a reduction by half of the already reduced number $\lfloor \frac{n}{2} \rfloor$ of constraints required for circulant MHMs, namely

$$\sum_{i=0}^{n-1} c_i c_{(i+j) \pmod{n}} = 0, j = 1, 2 ... \left\lfloor \frac{n}{2} \right\rfloor,$$

and it has been previously shown that the reduction in the circulant case facilitated the identification of direct solutions to produce circulant MHMs of all orders $n \neq 4$ [1]. Thus, this further reduction should facilitate finding direct solutions to produce mixed-circulant MHMs of even orders. We leave this as an open problem, however, and instead proceed with alternative ways of constructing mixed-circulant MHMs.

Algorithm 2. *Construct two order n MHMs A and B over alphabets $\mathcal{A} = \{a_0, \ldots, a_{n-1}\}$ and $\mathcal{B} = \{b_0, \ldots, b_{n-1}\}$, respectively. Then, define an order $2n$ mixed-circulant matrix C with alphabet $c_{2i} = a_i$ and $c_{2i-1} = b_i$, for $i = 0, 1, \ldots, n-1$.*

Theorem 3. *Any order $2n$ matrix C obtained via Algorithm 2 is a mixed-circulant MHM. Furthermore, for each order $2n$, it is possible to construct such a mixed-circulant MHM of full entry count.*

Proof. To verify that the matrix C defined via Algorithm 2 is a mixed-circulant MHM, it suffices to verify that

$$\sum_{i=0}^{2n-1} c_i c_{i+2j \pmod{2n}} = 0 \tag{10}$$

for $j = 1, 2, \ldots, \lfloor \frac{n}{2} \rfloor$. Note that for each j we have:

$$\begin{aligned} \sum_{i=0}^{2n-1} c_i c_{i+2j} &= \sum_{i=0}^{n-1} a_i a_{i+j \pmod{n}} + \sum_{i=0}^{n-1} b_i b_{i+n-j \pmod{n}} && (11) \\ &= 0 - 0 && (12) \\ &= 0 && (13) \end{aligned}$$

Eqn. (11) follows directly from the definition of C. Eqn. (12) follows as the summations of Eqn. (11) respectively represent inner products of columns 1 and $n-j$ of A (an MHM) and of B (another MHM).

Furthermore, to construct such a C with entry count $2n$, begin with two entry count n MHMs constructed as follows. For $n = 4$, let $A = H_4$ as defined in Eqn. (10) of [1], and let $B = 2H_4$. The resulting mixed-circulant MHM of order 8 has full entry count over the alphabet $\{1, 2, \ldots, 8\}$. For $n \neq 4$, let A be an entry count n circulant MHM generated by the construction in [1] over alphabet $\{a_0, a_1, \ldots, a_{n-1}\}$ where $a_i = r^i$ for $i = 0, 1, \ldots n-2$ and $a_{n-1} = -\frac{(r^{n-1}-r)}{r^2-1}$, and where all entries might be multiplied by a constant to ensure each entry is an integer given the chosen values of $r \neq 0$ and n. Although B can be chosen as a simple suitable scalar multiple of A, we suggest constructing B over an independent alphabet, specifically by generalizing the construction in [1] to produce a back-circulant MHM B. (This is done in anticipation of our use of circulant and back-circulant matrices to construct $\mathbb{C}$-MHMs in Subsection 3.2). In the circulant construction and its obvious back-circulant generalization, the ratio r in the alphabets can be chosen arbitrarily, so the alphabets for A and B can be chosen not to overlap. □

Example 2. *Let* $A = \begin{bmatrix} 3 & -14 & 24 & 12 & 6 \\ 6 & 3 & -14 & 24 & 12 \\ 12 & 6 & 3 & -14 & 24 \\ 24 & 12 & 6 & 3 & -14 \\ -14 & 24 & 12 & 6 & 3 \end{bmatrix}$ *be obtained via the construction in [1], and let* $B = \begin{bmatrix} 7 & 42 & 252 & 1512 & -258 \\ 42 & 252 & 1512 & -258 & 7 \\ 252 & 1512 & -258 & 7 & 42 \\ 1512 & -258 & 7 & 42 & 252 \\ -258 & 7 & 42 & 252 & 1512 \end{bmatrix}$ *be obtained via its obvious back-circulant generalization. Then, A and B are mixed and signed to create a full entry count order 10 mixed-circulant MHM C:*

$$C = \begin{bmatrix} 3 & 7 & -14 & 42 & 24 & 252 & 12 & 1512 & 6 & -258 \\ 7 & -3 & -258 & -6 & 1512 & -12 & 252 & -24 & 42 & 14 \\ 6 & -258 & 3 & 7 & -14 & 42 & 24 & 252 & 12 & 1512 \\ 42 & 14 & 7 & -3 & -258 & -6 & 1512 & -12 & 252 & -24 \\ 12 & 1512 & 6 & -258 & 3 & 7 & -14 & 42 & 24 & 252 \\ 252 & -24 & 42 & 14 & 7 & -3 & -258 & -6 & 1512 & -12 \\ 24 & 252 & 12 & 1512 & 6 & -258 & 3 & 7 & -14 & 42 \\ 1512 & -12 & 252 & -24 & 42 & 14 & 7 & -3 & -258 & -6 \\ -14 & 42 & 24 & 252 & 12 & 1512 & 6 & -258 & 3 & 7 \\ -258 & -6 & 1512 & -12 & 252 & -24 & 42 & 14 & 7 & -3 \end{bmatrix}$$

The advantage of this mixed-circulant construction over the existing circulant MHM construction [1] is that we can achieve an order $2n$ MHM whose entries do not follow a single geometric progression (with one additional entry), meaning it is possible to choose two shorter geometric progressions whose overall span (including the two additional entries) is smaller than possible using the currently known constructions for circulant MHMs of order $2n$ [1]. Although this does not provide as much control as the arithmetic progression construction of Subsection 2.2, this construction exists for all orders $2n$ while the arithmetic progression construction exists only for orders 2^n.

We now introduce c-MHMs as a natural extension of c-Hadamard matrices [10] and use them to produce mixed-circulant MHMs. These c-MHMs will be used again later to build $\mathbb{C}$-MHMs in Subsection 3.2.

Definition 6. *A* c-multilevel Hadamard matrix *(c-MHM) N of order n is an $n \times n$ matrix such that each row and column contains some permutation of the alphabet $\{a_0, a_1, ..., a_{n-1}\}$, up to sign and with entries not necessarily distinct, such that $NN^T = N^T N = \sum |a_i|^2 I + L_c$, where L_c is a uniform matrix of value c, with zeros down the main diagonal.*

Whereas above we used a pair of order n MHMs to build an order $2n$ mixed-circulant MHM, below we generalize this construction and use a pair of oppositely-signed order n c-MHMs (one with value c and the other with value $-c$) to build an order $2n$ mixed-circulant MMH.

Begin with a circulant matrix A' over alphabet $\{a, b, b, \ldots, b\}$ and a back-circulant matrix B' over alphabet $\{x, y, y, \ldots, y\}$, where a, b, x, y are chosen to satisfy $2ab + (n-2)b^2 = -2xy - (n-2)y^2$, so that the value of the inner product of distinct columns within A' is the negative of the value of the inner product of distinct columns within B'. Then, for $n \neq 4$, multiply both A' and B' on the left by the same full entry count order $n \neq 4$ circulant MHM M over alphabet $\mathcal{M}$ obtained using the construction in [1]. Since MHMs act as linear transformations that preserve inner products, the resulting matrix $A = MA'$ is an order n c-MHM, where $c = (2ab + (n-2)b^2) \sum_{i=0}^{n-1} m_i^2$, and similarly $B = MB'$ is an order n $(-c)$-MHM. Properties of circulant and back-circulant matrices imply that A is circulant and B is back-circulant. By choosing $a \neq b$ such that $a, b \notin \mathcal{M}$, we see that $A = MA'$ has entry count n, and similarly for B. Then, we can construct a mixed-circulant matrix C whose alphabet is defined by $c_{2j} = a_j$ and $c_{2j+1} = b_j$, for $j = 0, 1, \ldots, n-1$. The proof that C is indeed a mixed-circulant MHM follows similarly to the proof of Theorem 2, except that we replace Eqn. (13) with $c - c$, as A is now a c-MHM and B is now a $(-c)$-MHM.

Note that this construction of mixed-circulant MHMs also holds in the case of $n = 4$ by replacing M with the aforementioned full entry count order 4 MHM H_4. $A = H_4 A'$ and $B = H_4 B'$ will be c- and $(-c)$-MHMs, respectively, and it is possible to choose the alphabet entries so that each is of full entry count, but they will not be circulant and back-circulant, respectively. The circulance and back-circulance of A and B, respectively, do not play any role in verifying that they can be mixed and signed to form an order $2n$ mixed-circulant MHM. We only take A and B to be circulant and back-circulant, respectively, here in anticipation of our use of such pairs of c- and $(-c)$-MHMs in Subsection 3.2.

It is straightforward in most cases to ensure that the resulting mixed-circulant MHM is of full entry count. As demonstrated in the following example, the entries in the mixed-circulant MHMs constructed using c-MHMs and $(-c)$-MHMs are not simply intertwined geometric progressions (plus two additional values) as are the mixed-circulant MHMs constructed via Algorithm 2, so this construction provides some variation.

Example 3. *We now illustrate this generalization of the mixed-circulant construction using a value of* $c = 1,614,480$. *First, we build a circulant (-1,614,480)-MHM A and a back-circulant (1,614,480)-MHM B using appropriate circulant and back-circulant matrices A' and B' and an appropriate full entry count MHM M:*

$$A = MA'$$
$$= \begin{bmatrix} 3 & -14 & 24 & 12 & 6 \\ 6 & 3 & -14 & 24 & 12 \\ 12 & 6 & 3 & -14 & 24 \\ 24 & 12 & 6 & 3 & -14 \\ -14 & 24 & 12 & 6 & 3 \end{bmatrix} \begin{bmatrix} -117 & 8 & 8 & 8 & 8 \\ 8 & -117 & 8 & 8 & 8 \\ 8 & 8 & -117 & 8 & 8 \\ 8 & 8 & 8 & -117 & 8 \\ 8 & 8 & 8 & 8 & -117 \end{bmatrix}$$
$$= \begin{bmatrix} -127 & 1998 & -2752 & -1252 & -502 \\ -502 & -127 & 1998 & -2752 & -1252 \\ -1252 & -502 & -127 & 1998 & -2752 \\ -2752 & -1252 & -502 & -127 & 1998 \\ 1998 & -2752 & -1252 & -502 & -127 \end{bmatrix}$$

$$
\begin{aligned}
B &= MB' \\
&= \begin{bmatrix} 3 & -14 & 24 & 12 & 6 \\ 6 & 3 & -14 & 24 & 12 \\ 12 & 6 & 3 & -14 & 24 \\ 24 & 12 & 6 & 3 & -14 \\ -14 & 24 & 12 & 6 & 3 \end{bmatrix} \begin{bmatrix} 12 & 20 & 20 & 20 & 20 \\ 20 & 20 & 20 & 20 & 12 \\ 20 & 20 & 20 & 12 & 20 \\ 20 & 20 & 12 & 20 & 20 \\ 20 & 12 & 20 & 20 & 20 \end{bmatrix} \\
&= \begin{bmatrix} 596 & 572 & 524 & 428 & 732 \\ 572 & 524 & 428 & 732 & 596 \\ 524 & 428 & 732 & 596 & 572 \\ 428 & 732 & 596 & 572 & 524 \\ 732 & 596 & 572 & 524 & 428 \end{bmatrix}
\end{aligned}
$$

We then combine A and B to form a full entry count order 10 mixed-circulant MHM C:

$$
C = \begin{bmatrix}
-127 & 596 & 1998 & 572 & -2752 & 524 & -1252 & 428 & -502 & 732 \\
596 & 127 & 732 & 502 & 428 & 1252 & 524 & 2752 & 572 & -1998 \\
-502 & 732 & -127 & 596 & 1998 & 572 & -2752 & 524 & -1252 & 428 \\
572 & -1998 & 596 & 127 & 732 & 502 & 428 & 1252 & 524 & 2752 \\
-1252 & 428 & -502 & 732 & -127 & 596 & 1998 & 572 & -2752 & 524 \\
524 & 2752 & 572 & -1998 & 596 & 127 & 732 & 502 & 428 & 1252 \\
-2752 & 524 & -1252 & 428 & -502 & 732 & -127 & 596 & 1998 & 572 \\
428 & 1252 & 524 & 2752 & 572 & -1998 & 596 & 127 & 732 & 502 \\
1998 & 572 & -2752 & 524 & -1252 & 428 & -502 & 732 & -127 & 596 \\
732 & 502 & 428 & 1252 & 524 & 2752 & 572 & -1998 & 596 & 127
\end{bmatrix}
$$

3 MHMs over the Gaussian Integers: Complex $\mathbb{C}$-MHMs

3.1 Introduction to $\mathbb{C}$-MHMs

We introduce complex $\mathbb{C}$-MHMs motivated by potential future applications similar to those of Hadamard matrices [15], real MHMs [17], and/or complex orthogonal designs [11, 18]. Our work here focuses on building theory and constructions for these $\mathbb{C}$-MHMs.

Definition 7. *Given two $n \times n$ matrices A and B with integer alphabets $\mathcal{A} = \{a_0, a_1, \ldots, a_{n-1}\}$ and $\mathcal{B} = \{b_0, b_1, \ldots, b_{n-1}\}$, respectively, such that each row and column of A and B contain some permutation of their respective alphabet up to sign and where the entries of the alphabets are not necessarily distinct, then $Z = A + Bi$ is an order n* $\mathbb{C}$-multilevel Hadamard matrix *($\mathbb{C}$-*

MHM) if $ZZ^H = \sum_{i=0}^{n-1}(a_i^2 + b_i^2)I$*, where* H *denotes the Hermitian transpose. The complex entries of* Z *can be written as* $z_h = \pm a_j \pm b_k i$*, for some* $a_j \in \mathcal{A}$ *and some* $b_k \in \mathcal{B}$*.*

All results can be trivially generalized if we instead require $Z^H Z = \sum_{i=0}^{n-1}(a_i^2 + b_i^2)I$, which would also be the constraint used in the case of future rectangular $\mathbb{C}$-MHMs.

In analogy to the entry count of an MHM, we define the following:

Definition 8. *The* entry count triple *of a* $\mathbb{C}$*-MHM* $Z = A + Bi$ *is an ordered triple* (s, t, r) *such that* s *is the number of distinct entries* a *in* A *up to sign,* t *is the number of distinct entries* b *in* B *up to sign, and* r *is the number of distinct entries* $a + bi$ *in* Z *up to sign in each component.*

We propose that it is of primary interest to find $\mathbb{C}$-MHMs with the following entry count triples: (n, n, n^2) and (n, n, n). In the former case, we have maximized the amount of information possible in the real and imaginary dimensions, and we have mixed the real and imaginary components maximally so that every pairing occurs precisely once. In the later case, we have again maximized the number of distinct entries in the real and imaginary dimensions, but we have now aligned the alphabets so that the complex combinations behave more like the complex variables in a complex orthogonal design: there are n complex pairings (up to sign in each component) that are repeated once per row and per column. These are the extremal cases, perhaps the most interesting to achieve from a mathematical perspective. They are also perhaps the most promising from an applications perspective, as they allow for maximal mixing of terms or maximum repetition of terms. While either extreme is likely to be useful in applications, we note that these entry count triples do not reflect all possibly useful properties of a $\mathbb{C}$-MHM. For example, a $\mathbb{C}$-MHM could trivially achieve the desirable entry count triple (n, n, n) by taking $A = B$ in the definition; this is less desirable than achieving (n, n, n) via matrices A and B on independent alphabets.

In order to construct $\mathbb{C}$-MHMs, we must examine the constraints implied by Definition 7. Formally, we need

$$ZZ^H = (A + Bi)(A + Bi)^H = AA^T - AB^T i + BA^T i + BB^T = rI,$$

where r is the real number $\sum_{i=0}^{n-1}(a_i^2 + b_i^2)$. Because r is real, the following equations must be satisfied

$$AA^T + BB^T = rI \tag{14}$$

$$AB^T = BA^T \tag{15}$$

In the following subsections, we offer several constructions that simultaneously fulfill Equations (14) and (15) while specifically achieving entry count triple (n, n, n^2) for odd n and entry count triple (n, n, n) for all n.

3.2 $\mathbb{C}$-MHMs with Entry Count Triple (n, n, n^2)

We note first that Eqn. (14) can be satisfied by requiring A and B to be MHMs. Next, we note that pairs of matrices A and B that satisfy Eqn. (15) are precisely known as *amicable* matrices, as previously introduced in the context of real and complex orthogonal designs [12, 20, 21] and in the context of Hadamard matrices [19]. Thus, we present a construction based on amicable pairs of MHMs. This construction was inspired by Seberry (Wallis)'s work on amicable Hadamard matrices [19] and by Adams *et al.*'s previous work on circulant MHMs [1]. Recall the following result (Theorem 1 from [19]) which can be restated as:

Lemma 2. *[19] If A is a circulant $n \times n$ matrix and B is a back-circulant $n \times n$ matrix, then A and B are amicable.*

This allows us to obtain our first $\mathbb{C}$-MHM construction:

Theorem 4. *For any odd order n, let A be an order n circulant MHM and B be an order n back-circulant MHM such that A and B have disjoint alphabets of entry count n. Then, $Z = A + Bi$ is a $\mathbb{C}$-MHM with entry count triple (n, n, n^2).*

Proof. First, recall that circulant order n MHMs of entry count n exist for all orders $n \neq 4$ using the construction provided by Adams *et al.* [1], and the construction there can be modified in an obvious manner to provide back-circulant order n MHMs of entry count n for all orders $n \neq 4$ as well. The construction provides enough choice in alphabet composition to ensure that A and B can be formed using disjoint alphabets, as it uses geometric progressions with arbitrary initial terms and arbitrary common ratios (plus an additional term in each alphabet). Now, for n odd, let A be a circulant MHM and B be a back-circulant MHM with disjoint alphabets of entry count n. Eqn. (14) is satisfied since A and B are MHMs, and Eqn. (15) is satisfied by Lemma 2. Thus, $A + Bi$ is a complex MHM.

Since A and B both have entry count n, the first two entry counts of the entry count triple of $A+Bi$ are n. Now, if we restrict to odd n, the third entry count value of $A+Bi$ is n^2; to see this, note that if two entries (i_1, j_1) and (i_2, j_2) were equal it would imply

$$\begin{aligned}
a_{i_1-j_1 \pmod n} &= a_{i_2-j_2 \pmod n} && (16)\\
b_{i_1+j_1 \pmod n} &= b_{i_2+j_2 \pmod n} && (17)\\
i_1-j_1 \pmod n &\equiv i_2-j_2 \pmod n && (18)\\
i_1+j_1 \pmod n &\equiv i_2+j_2 \pmod n && (19)\\
2(i_1-i_2) &\equiv 0 \pmod n && (20)\\
2(j_1-j_2) &\equiv 0 \pmod n && (21)\\
i_1 &= i_2 && (22)\\
j_1 &= j_2 && (23)
\end{aligned}$$

Eqn. (18) follows from Eqn. (16) because A is an MHM with entry count n. Similarly, since B is a MHM with entry count n, Eqn. (19) follows from (17). Add and subtract Equations (17) and (19) to get Equations (20) and (21), respectively. Since $gcd(2, n) = 1$, Equations (22) and (23) follow from (20) and (21), respectively. □

This construction implies the following concise result:

Corollary 2. *For any odd n, there exists an order n $\mathbb{C}$-MHM with entry count triple (n, n, n^2).*

Example 4. *The order 5 component MHMs from Example 2 can be combined to form an order 5 $\mathbb{C}$-MHM with entry count triple (5, 5, 25).*

$$A+Bi = \begin{bmatrix} 3+7i & -14+42i & 24+252i & 12+1512i & 6-258i \\ 6+42i & 3+252i & -14+1512i & 24-258i & 12+7i \\ 12+252i & 6+1512i & 3-258i & -14+7i & 24+42i \\ 24+1512i & 12-258i & 6+7i & 3+42i & -14+252i \\ -14-258i & 24+7i & 12+42i & 6+252i & 3+1512i \end{bmatrix}$$

We can now generalize this construction using c-MHMs: for n odd, take A to be a circulant c-MHM and B to be a back-circulant $(-c)$-MHM, whose existence was developed in Subsection 2.3. It is straight-forward to verify that Eqn. (14) is satisfied as $L_c + L_{-c} = 0$, and Eqn. (15) is satisfied as Lemma 2 implies that A and B are amicable. Thus, $Z = A + Bi$ is a $\mathbb{C}$-MHM.

Example 5. *Let A the circulant (-1,614,480)-MHM and B be the back-circulant (1,614,480)-MHM from Example 3. Then $A + Bi$ is an order 5 $\mathbb{C}$-MHM with entry count triple (5, 5, 25):*

$$\begin{bmatrix} -127+596i & 1998+572i & -2752+524i & -1252+428i & -502+732i \\ -502+572i & -127+524i & 1998+428i & -2752+732i & -1252+596i \\ -1252+524i & -502+428i & -127+732i & 1998+596i & -2752+572i \\ -2752+428i & -1252+732i & -502+596i & -127+572i & 1998+524i \\ 1998+732i & -2752+596i & -1252+572i & -502+524i & -127+428i \end{bmatrix}$$

3.3 $\mathbb{C}$-MHMs with Entry Count Triple (n, n, n)

We begin with a trivial solution to the problem of finding $\mathbb{C}$-MHMs with entry count triple (n, n, n) by taking A to be a full entry count MHM and by taking B to be a nonzero scalar multiple of A; say $B = cA$ where c is some integer. Then, A and B are both MHMs, so Eqn. (14) is satisfied. Eqn. (15) is also satisfied since $AB^T = A(cA)^T = (cA)A^T = BA^T$. It follows that $Z = A + Bi$ is a complex MHM. This construction provides $\mathbb{C}$-MHMs of entry count triple (n, n, n) as a required full entry count MHM A can be constructed for any order n using prior constructions [1], and then if $c \neq 0$, Z must be of entry count (n, n, n), as A and B would have the same pattern distribution.

Our next and final construction of $\mathbb{C}$-MHMs with entry count triple (n, n, n) builds on the arithmetic progression construction of MHMs given in Algorithm 1 in Subsection 2.2. In addition to achieving entry count (n, n, n), this construction allows the most control over the span and spacing of the entries. However, this construction only applies for orders $n = 2^m$.

We first provide a new algorithm for achieving MHMs of order 2^m over arithmetic progression alphabets. This new algorithm produces MHMs Y_m that closely resemble the MHMs X_m of Algorithm 1; effectively, MHMs formed via Algorithm 3 can also be made by reversing the order in which the columns appear in an MHM constructed via Algorithm 1.

Algorithm 3. *Given two positive integers b and e, Let* $Y_1 = \begin{bmatrix} b+e & b \\ -b & b+e \end{bmatrix}$, *and for $k > 1$, let*

$$Y_{k+1} = \begin{bmatrix} Y_k + e2^k U_k & Y_k \\ Y_k J_k & -Y_k J_k - e2^k U_k J_k \end{bmatrix},$$

where U_k is the sign matrix of Y_k and J_k is half-identity matrix of order 2^k.

A proof that Algorithm 3 generates full entry count MHMs of all orders 2^m over arithmetic progression alphabets follows similarly to the analogous proof for Algorithm 1. As with Algorithm 1, we can modify Algorithm 3 to allow for more general alphabets than strict arithmetic progressions by using different integers e_k at each iteration, rather than using $e2^k$ at each iteration.

Below, we will prove that if X and Y are MHMs of order 2^m constructed via Algorithms 1 and 3 respectively, then $X + Yi$ is a $\mathbb{C}$-MHM of order 2^m. To show this, the following lemma will be needed; proofs for each part are provided in Appendix B.

Lemma 3. *Suppose that X_m is an order 2^m MHM constructed via Algorithm 1 using a common difference of d and having a sign matrix S_m; suppose that Y_m is an order 2^m MHM constructed via Algorithm 3 using a common difference of e and having a sign matrix U_m; and suppose J_m is the half-identity matrix of order 2^m. Then*

1. $S_m J_m U_m^T$ *is skew-symmetric;*
2. $X_m U_m^T$ *is symmetric;*
3. $S_m Y_m^T$ *is symmetric; and*
4. $eX_m J_m U_m^T + dS_m J_m Y_m^T$ *is skew-symmetric.*

Theorem 5. *Suppose that X_m is an order 2^m MHM constructed via Algorithm 1 and Y_m is an order 2^m MHM constructed via Algorithm 3. Then $Z_m = X_m + Y_m i$ is an order 2^m $\mathbb{C}$-MHM of entry count $(2^m, 2^m, 2^m)$, and the alphabets of X_m and Y_m are independent arithmetic progressions.*

Proof. As X_m and Y_m are MHMs, it suffices to prove that X_m and Y_m are amicable. We must show $X_m Y_m^T = Y_m X_m^T$, or equivalently that $X_m Y_m^T$ is symmetric. We proceed by induction on m.

Consider a base case when $k = 1$:

$$\begin{aligned} X_1 Y_1^T &= \begin{bmatrix} x & x+d \\ x+d & -x \end{bmatrix} \begin{bmatrix} y+e & -y \\ y & y+e \end{bmatrix} \\ &= \begin{bmatrix} 2xy + ex + dy & de + dy + ex \\ de + dy + ex & -2xy - dy - ex \end{bmatrix} \end{aligned}$$

is clearly symmetric.

Assume $X_k Y_k^T = Y_k^T X_k$ for some $k \geq 1$. Then, as defined in Algorithms 1 and 3, we have $X_{k+1} = \begin{bmatrix} X_k & X_k + d2^k S_k \\ X_k J_k + d2^k S_k J_k & -X_k J_k \end{bmatrix}$, and $Y_{k+1} = \begin{bmatrix} Y_k + e2^k U_k & Y_k \\ Y_k J_k & -Y_k J_k - e2^k U_k J_k \end{bmatrix}$, so that $X_{k+1} Y_{k+1}^T =$

$$\begin{bmatrix} 2X_k Y_k^T + e2^k X_k U_k^T + d2^k S_k Y_k^T & -e2^k X_k J_k U_k^T - d2^k S_k J_k Y_k^T - ed2^{2k} S_k J_k U_k^T \\ +e2^k X_k J_k U_k^T + d2^k S_k J_k Y_k^T + ed2^{2k} S_k J_k U_k^T & 2X_k Y_k^T + e2^k X_k U_k^T + d2^k S_k Y_k^T \end{bmatrix}$$

It now follows directly from Lemma 3 and the inductive hypothesis that $X_{k+1}Y_{k+1}^T$ is symmetric. Therefore, by the principle of mathematical induction, X_m and Y_m are amicable for all m, and so $X_m + Y_m i$ is a $\mathbb{C}$-MHM for all m.

It is clear that X_m and Y_m each have entry count n, as their alphabets are independently constructed arithmetic progressions (up to sign). The construction then implies entry count triple (n, n, n). □

Example 6. *In this example of an order 8 $\mathbb{C}$-MHM with entry count triple (8, 8, 8) constructed using Theorem 5, the first component matrix is an extension of Example 1 with initial value 3 and common difference 2. The second component matrix is formed using Algorithm 3 with initial value 4 and common difference 3.*

$$\begin{bmatrix}
3+25i & 5+22i & 7+19i & 9+16i & 11+13i & 13+10i & 15+7i & 17+4i \\
5-22i & -3+25i & 9-16i & -7+19i & 13-10i & -11+13i & 17-4i & -15+7i \\
7+19i & -9-16i & -3-25i & 5+22i & 15+7i & -17-4i & -11-13i & 13+10i \\
9-16i & 7-19i & -5+22i & -3+25i & 17-4i & 15-7i & -13+10i & -11+13i \\
11+13i & 13+10i & -15-7i & -17-4i & -3-25i & -5-22i & 7+19i & 9+16i \\
13-10i & -11+13i & -17+4i & 15-7i & -5+22i & 3-25i & 9-16i & -7+19i \\
15+7i & -17-4i & 11+13i & -13-10i & -7-19i & 9+16i & -3-25i & 5+22i \\
17-4i & 15-7i & 13-10i & 11-13i & -9+16i & -7+19i & -5+22i & -3+25i
\end{bmatrix}$$

This is potentially our most interesting construction for $\mathbb{C}$-MHMs, as we have complete control over the spacing and span of the two component alphabets. The other constructions of $\mathbb{C}$-MHMs exist for more general sizes, however.

The trivial scalar multiple construction and Theorem 5 together imply the following concise result:

Corollary 3. *For any given order n, there exists a complex $\mathbb{C}$-MHM with entry count triple (n, n, n).*

4 MHMs over the Hamiltonian Integers: Quaternion $\mathbb{Q}$-MHMs

As the next natural generalization, we define MHMs over the Hamiltonian integers. Such matrices may have similar applications as those enjoyed by Hadamard matrices [15], real MHMs [17], and/or quaternion orthogonal designs [2, 14, 22].

Definition 9. *Given four $n \times n$ matrices A, B, C, and D with integer alphabets $\mathcal{A} = \{a_0, a_1, \ldots, a_{n-1}\}$, $\mathcal{B} = \{b_0, b_1, \ldots, b_{n-1}\}$, $\mathcal{C} = \{c_0, c_1, \ldots, c_{n-1}\}$, and $\mathcal{D} = \{d_0, d_1, \ldots, d_{n-1}\}$ such that each row and column of A, B, C, D contain some permutation of their respective alphabet up to sign, and where the entries of the alphabets are not necessarily distinct, then $H = A + Bi + Cj + Dk$ is an order n $\mathbb{Q}$-*multilevel Hadamard matrix *($\mathbb{Q}$-MHM) if $HH^Q = \sum (a_i^2 + b_i^2 + c_i^2 + d_i^2)I$, where Q denotes the quaternion conjugate transpose. The quaternion entries of H can be written as $q_h = \pm a_j \pm b_k i \pm c_l j \pm d_m k$, for some $a_j \in \mathcal{A}$, $b_k \in \mathcal{B}$, $c_l \in \mathcal{C}$, and $d_m \in \mathcal{D}$.*

All results can be trivially generalized if we instead require $H^Q H = \sum (a_i^2 + b_i^2 + c_i^2 + d_i^2)I$, which would also be the constraint used in the case of future rectangular $\mathbb{Q}$-MHMs. Naturally, we have the following definition:

Definition 10. *The* entry count quintuple *of a $\mathbb{Q}$-MHM $H = A + Bi + Cj + Dk$ is an ordered quintuple (s, t, u, v, r) such that s is the number of distinct entries a in A up to sign, t is the number of distinct entries b in B up to sign, u is the number of distinct entries c in C up to sign, v is the number of distinct entries d in D up to sign, and r is the number of distinct entries $a + bi + cj + dk$ in H up to sign in each component.*

With similar motivation as in the complex case, we propose that it is of primary interest to build order n $\mathbb{Q}$-MHMs with entry count quintuples (n, n, n, n, n) and (n, n, n, n, n^2). In the former case, there are n entries of distinct absolute value in each of the alphabets of the four component matrices, and these entries align to form n quaternion integers (up to sign in each component) that each appear in every row/column of the $\mathbb{Q}$-MHM. In the latter case, the four component alphabets are mixed so that no 4-tuple appears more than once in the $\mathbb{Q}$-MHM. Suggested future work involves investigating the existence of higher-dimensional $\mathbb{Q}$-MHMs, defined analogously to higher-dimensional Hadamard matrices, real MHMs, and RODs [1, 6, 7, 8, 9, 13, 16, 23], motivated by the search for higher-dimensional $\mathbb{Q}$-MHMs that include all possible n^4 quaternion entries (up to sign in each component) given n distinct entries in each of the four component alphabets.

We first consider the conditions that must be true on the component matrices A, B, C, and D in order to build a $\mathbb{Q}$-MHM. According to Definition 9, we must have $(A + Bi + Cj + Dk)(A + Bi + Cj + Dk)^Q = rI$ where $r = \sum (a_i^2 + b_i^2 + c_i^2 + d_i^2)$. When expanded and grouped according to quaternion parts, this implies the following equations must be satisfied:

$$AA^T + BB^T + CC^T + DD^T = rI \tag{24}$$
$$BA^T - AB^T + DC^T - CD^T = 0 \tag{25}$$
$$CA^T - AC^T + BD^T - DB^T = 0 \tag{26}$$
$$DA^T - AD^T + CB^T - BC^T = 0 \tag{27}$$

One way to satisfy these equations would be to produce four pairwise amicable MHMs A, B, C, and D. It remains an open problem to investigate whether such families of MHMs exist for nontrivial parameters. We instead provide alternative ways to build $\mathbb{Q}$-MHMs.

A simple way to satisfy the governing equations for A, B, C, D is to choose any two of these matrices to be the component matrices of a $\mathbb{C}$-MHM constructed in Section 3, and then let the other component matrices be integer multiples of these, respectively. For example, given a $\mathbb{C}$-MHM with component matrices A, B and two nonzero integers α, β, we can form a $\mathbb{Q}$-MHM of the form $A+Bi+\alpha Aj+\beta Bk$. This allows us to find $\mathbb{Q}$-MHMs of entry count quintuple (n, n, n, n, n^2) and (n, n, n, n, n), however the resulting examples are not entirely satisfying in that two of the component matrices must be scalar multiples of the other two, respectively.

To illustrate, consider the construction in Subsection 3.2 that uses amicable pairs of circulant and back-circulant MHMs. Suppose, for example, that A and C are amicable MHMs so that $AC^T = CA^T$, and then define $B = \alpha A$ and $D = \beta C$, where α, β are nonzero integers. Then, Eqn. (24) holds as A, B, C, D are all MHMs. Eqn. (25) holds because $BA^T = (\alpha A)A^T = A(\alpha A)^T = AB^T$, so A and B are amicable, and similarly with C and D. Eqn. (26) holds, because A and C are given as amicable and thus $BD^T = (\alpha A)(\beta C)^T = \alpha\beta CA^T = DB^T$, so B and D are also amicable. It is similar to show A, D are amicable and B, C are amicable, thus satisfying Eqn. (27). Thus, it follows from our work in Subsection 3.2, that if we restrict ourselves to odd orders n and take A to be a full entry count circulant MHM of order n and C to be a full entry count back-circulant MHM of order n (using the construction from [1] and its obvious back-circulant generalization), then $H = A + \alpha Ai + Cj + \beta Ck$ is a $\mathbb{Q}$-MHM of entry count quintuple (n, n, n, n, n^2). Similarly, we can satisfy the defining equations for $\mathbb{Q}$-MHMs if we take one of the component matrices to be a circulant c-MHM, another to be a back-circulant $(-c)$-MHM, (such pairs are developed in Subsection 2.3) and the others to be nonzero scalar multiples of these, respectively.

We can trivially obtain the entry count quintuple (n, n, n, n, n) by taking the four component matrices to be scalar multiples of the same matrix.

More substantially, we can take A (for example) to be a full entry count MHM generated via Algorithm 1 over an arithmetic progression alphabet and C (for example) as an MHM generated via Algorithm 3 over an independent arithmetic progression alphabet. Then, $H = A + \alpha Ai + Cj + \beta Ck$ is a $\mathbb{Q}$-MHM of entry count quintuple (n, n, n, n, n), where n is of the form 2^m.

Thus, we can obtain the desirable entry count quintuples of (n, n, n, n, n) and (n, n, n, n, n^2) by using generalizations of the complex constructions from Section 3 and the component pieces developed in Section 2. It remains open to find constructions that do not require two of the component matrices to be integer multiples of the others, respectively.

We provide one representative example of a $\mathbb{Q}$-MHM built using two component matrices of a $\mathbb{C}$-MHM and integer multiples thereof.

Example 7. *Let $A = X_2$ from Example 1 obtained via Algorithm 1, and let* $C = \begin{bmatrix} 2 & 4 & 6 & 8 \\ -4 & 2 & -8 & 6 \\ 6 & -8 & -2 & 4 \\ -8 & -6 & 4 & 2 \end{bmatrix}$ *be an MHM obtained via Algorithm 3. Then by Lemma 2, A and C are amicable. Letting $B = 2A$ and $D = 5C$ gives the following order 4 $\mathbb{Q}$-MHM with entry count quintuple (4, 4, 4, 4, 4):*

$$\begin{bmatrix} 3+6i+2j+10k & 5+10i+4j+20k & 7+14i+6j+30k & 9+18i+8j+40k \\ 5+10i-4j-20k & -3-6i+2j+10k & 9+18i-8j-40k & -7-14i+6j+30k \\ 7+14i+6j+30k & -9-18i-8j-40k & -3-6i-2j-10k & 5+10i+4j+20k \\ 9+18i-8j-40k & 7+14i-6j-30k & -5-10i+4j+20k & -3-6i+2j+10k \end{bmatrix}$$

5 Conclusions and Open Problems

This paper has explored multilevel Hadamard matrices over real, complex, and quaternion integers. We have shown a construction that produces real, full entry count, order 2^m MHMs over arithmetic progressions, allowing for evenly spaced and nicely bounded alphabets. Their alphabets make them preferable to the only other known MHMs that achieve full entry count for all orders n, as those MHMs have alphabets that contain geometric sequences [1]. This construction also implies the existence of full entry count order 2^m MHMs whose alphabets contain all prime numbers, which may be useful in future applications. We also presented constructions of mixed-circulant MHMs of order $2n$ using pairs of MHMs or c-MHMs of order n. We suggested an open problem of determining direct solutions for mixed-circulant MHMs and explained why we believe this to be a feasible problem.

We defined complex $\mathbb{C}$-MHMs and demonstrated several different construction techniques, thus achieving the proposed-as-desirable entry count triples (n, n, n^2) and (n, n, n). Most notably, one of the constructions achieving entry count triple (n, n, n) contains Gaussian integers such that the real components form an arithmetic progression and the imaginary components form an independent arithmetic progression.

Finally, we defined quaternion $\mathbb{Q}$-MHMs and generalized our $\mathbb{C}$-MHM constructions to build these $\mathbb{Q}$-MHMs. Our constructions allow us to achieve the proposed-as-desirable entry count quintuples (n, n, n, n, n^2) and (n, n, n, n, n), however the constructions do not allow for four completely independent alphabets. Our constructions allow for at most two independent alphabets, while the other alphabets are scalar multiples of these. We hope these preliminary constructions will inspire future work in constructing $\mathbb{Q}$-MHMs over alphabets that have fewer dependencies. We also suggest future work in studying higher-dimensional $\mathbb{Q}$-MHMs (defined analogously to the previously defined higher-dimensional Hadamard matrices, MHMs, and RODs) that include all possible n^4 quaternion entries (up to sign in each component) given n distinct entries in each of the four component alphabets. It has not yet been studied whether such structures exist for non-trivial parameters.

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Appendix A

This appendix contains proofs of the three parts of Lemma 1 from Subsection 2.2.

Lemma 1: Let X_m be an order 2^m matrix constructed via Algorithm 1, S_m be the corresponding sign matrix, and J_m be the half-identity matrix of order 2^m. Then,

1. S_m is a traditional Hadamard matrix;
2. $S_m X_m^T$ is skew; and

3. $X_m J_m S_m^T$ is symmetric.

Proof. (Part 1 by Induction) When $m = 1$,

$$S_1^T S_1 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = 2I_2$$

so S_1 is a Hadamard matrix and the base case holds. Suppose that S_k is a Hadamard matrix and thus $S_k S_k^T = S_k^T S_k = (2^k)I$ for some $k \geq 1$. Based on the construction of X_{k+1}, $S_{k+1} = \begin{bmatrix} S_k & S_k \\ S_k J_k & -S_k J_k \end{bmatrix}$, and so

$$\begin{aligned} S_{k+1}^T S_{k+1} &= \begin{bmatrix} S_k^T & J_k S_k^T \\ S_k^T & -J_k S_k^T \end{bmatrix} \begin{bmatrix} S_k & S_k \\ S_k J_k & -S_k J_k \end{bmatrix} \\ &= \begin{bmatrix} S_k^T S_k + J_k S_k^T S_k J_k & S_k^T S_k - J_k S_k^T S_k J_k \\ S_k^T S_k - J_k S_k^T S_k J_k & S_k^T S_k + J_k S_k^T S_k J_k \end{bmatrix} \end{aligned}$$

Then, as $S_k^T S_k = (2^k)I_{2^k}$ by our inductive hypothesis and as $J_k J_k = I_{2^k}$, we simplify to get

$$\begin{aligned} S_{k+1}^T S_{k+1} &= \begin{bmatrix} (2^k)I_{2^k} + (2^k)I_{2^k} & (2^k)I_{2^k} - (2^k)I_{2^k} \\ (2^k)I_{2^k} - (2^k)I_{2^k} & (2^k)I_{2^k} + (2^k)I_{2^k} \end{bmatrix} \\ &= 2^{k+1} \begin{bmatrix} I_{2^k} & 0 \\ 0 & I_{2^k} \end{bmatrix} \\ &= 2^{k+1} I_{2^{k+1}} \end{aligned}$$

A similar argument can be used to show that $S_{k+1} S_{k+1}^T = 2^{k+1} I_{2^{k+1}}$ and thus that S_{k+1} is Hadamard. By induction, S_m is Hadamard for all $m \geq 1$. □

Proof. (Part 2 by Induction) Consider the base case for $k = 1$:

$$S_1 X_1^T = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} a & a+d \\ a+d & -a \end{bmatrix} = \begin{bmatrix} 2a+d & d \\ -d & 2a+d \end{bmatrix}$$

is clearly skew. Suppose that $S_k X_k^T$ is skew for some $k \geq 1$. Then, consider

$$\begin{aligned} S_{k+1} X_{k+1}^T &= \begin{bmatrix} S_k & S_k \\ S_k J_k & -S_k J_k \end{bmatrix} \begin{bmatrix} X_k^T & d2^k J_k S_k^T + J_k X_k^T \\ X_k^T + d2^k S_k^T & -J_k X^T \end{bmatrix} \\ &= \begin{bmatrix} 2S_k X_k^T + d2^{k+1} I_{2^k} & d2^k S_k J_k S_k^T \\ -d2^k S_k J_k S_k^T & 2S_k X_k^T + d2^{k+1} I_{2^k} \end{bmatrix}. \end{aligned}$$

To show that this is skew, we need to first confirm that the transpose of the upper right block matrix plus the lower left block matrix gives 0 (equivalently the transpose of the lower left plus the upper right gives 0), which is obvious. We next need to confirm that the upper left block matrix plus its transpose (and the lower right block plus its transpose) gives a scalar multiple of the identity. This follows from our inductive hypothesis that $S_k X_k^T$ is skew. Thus, by the principle of mathematical induction, $S_m X_m^T$ is skew for all $m \geq 1$. □

Proof. (Part 3 by Induction) Consider the base case of $k = 1$:

$$X_1 J_1 S_1^T = (X_1 J_1) S_1^T = \begin{bmatrix} a & -(a+d) \\ a+d & a \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$
$$= \begin{bmatrix} -d & 2a+d \\ 2a+d & d \end{bmatrix}$$

is clearly symmetric. Assume that $X_k J_k S_k^T$ is symmetric for some $k \geq 1$. Then,

$$\begin{aligned} X_{k+1} J_{k+1} S_{k+1}^T &= (X_{k+1} J_{k+1}) S_{k+1}^T \\ &= \begin{bmatrix} X_k & -X_k - (d2^k) S_k \\ X_k J_k + (d2^k) S_k J_k & X_k J_k \end{bmatrix} \begin{bmatrix} S_k^T & J_k S_k^T \\ S_k^T & -J_k S_k^T \end{bmatrix} \\ &= \begin{bmatrix} -d2^{k+1} I & 2X_k J_k S_k^T + d2^k S_k J_k S_k^T \\ 2X_k J_k S_k^T + d2^k S_k J_k S_k^T & d2^{k+1} I \end{bmatrix}. \end{aligned}$$

Since $X_k J_k S_k^T$ is symmetric by the inductive hypothesis, $X_{k+1} J_{k+1} S_{k+1}^T$ is also symmetric. Thus, the result holds by induction. □

Appendix B

This appendix contains proofs of the four parts of Lemma 3 from Subsection 3.3.

Lemma 3 Suppose that X_m is an order 2^m MHM constructed via Algorithm 1 using a common difference of d and having a sign matrix S_m; suppose that Y_m is an order 2^m MHM constructed via Algorithm 3 using a common difference of e and having a sign matrix U_m; and suppose J_m is the half-identity matrix of order 2^m. Then

1. $S_m J_m U_m^T$ is skew-symmetric;
2. $X_m U_m^T$ is symmetric;
3. $S_m Y_m^T$ is symmetric; and

4. $eX_mJ_mU_m^T + dS_mJ_mY_m^T$ is skew-symmetric.

Proof. *(Part 1 by Induction)* As a base case, note that when $k = 1$,

$$S_1J_1U_1^T = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix}$$

is clearly skew-symmetric. Assume $S_kJ_kU_k^T = -(S_kJ_kU_k^T)^T$ for some $k \geq 1$, and now consider the following:

$$\begin{aligned} S_{k+1}J_{k+1}U_{k+1}^T &= \begin{bmatrix} S_k & S_k \\ S_kJ_k & -S_kJ_k \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix} \begin{bmatrix} U_k^T & J_kU_k^T \\ U_k^T & -J_kU_k^T \end{bmatrix} \\ &= \begin{bmatrix} S_k & S_k \\ S_kJ_k & -S_kJ_k \end{bmatrix} \begin{bmatrix} U_k^T & J_kU_k^T \\ -U_k^T & J_kU_k^T \end{bmatrix} \\ &= \begin{bmatrix} 0 & 2S_kJ_kU_k^T \\ 2S_kJ_kU_k^T & 0 \end{bmatrix}, \end{aligned}$$

and then by our inductive hypothesis, we have

$$(S_{k+1}J_{k+1}U_{k+1}^T)^T = \begin{bmatrix} 0 & 2(S_kJ_kU_k^T)^T \\ 2(S_kJ_kU_k^T)^T & 0 \end{bmatrix} = \begin{bmatrix} 0 & -2S_kJ_kU_k^T \\ -2S_kJ_kU_k^T & 0 \end{bmatrix}$$

and thus $S_{k+1}J_{k+1}U_{k+1}^T = -(S_{k+1}J_{k+1}U_k^T)^T$. It now follows by induction that $S_mJ_mU_m^T$ is skew-symmetric for all $m \geq 1$. □

Proof. *(Part 2 by Induction)* Consider the base case $k = 1$:

$$\begin{aligned} X_1U_1^T &= \begin{bmatrix} a & a+d \\ a+d & -a \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 2a+d & d \\ d & -2a-d \end{bmatrix}, \end{aligned}$$

which is clearly symmetric.

Assume that $X_kU_k^T = (X_kU_k^T)^T$ for some $k \geq 1$, and then consider:

$$\begin{aligned} X_{k+1}U_{k+1}^T &= \begin{bmatrix} X_k & X_k + d2^kS_k \\ X_kJ_k + d2^kS_kJ_k & -X_kJ_k \end{bmatrix} \begin{bmatrix} U_k^T & J_kU_k^T \\ U_k^T & -J_kU_k^T \end{bmatrix} \\ &= \begin{bmatrix} 2X_kU_k^T + d2^kS_kUk^T & -d2^kS_kJ_kU_k^T \\ d2^kS_kJ_kU_k^T & 2X_kU_k^T + d2^kS_kU_k^T \end{bmatrix} \end{aligned}$$

This implies

$$(X_{k+1}U_{k+1}^T)^T = \left[\begin{array}{cc} 2(X_kU_k^T)^T + d2^k(S_kUk^T)^T & d2^k(S_kJ_kU_k^T)^T \\ -d2^k(S_kJ_kU_k^T)^T & 2(X_kU_k^T)^T + d2^k(S_kU_k^T)^T \end{array} \right] \quad (28)$$

$$= \left[\begin{array}{cc} 2X_kU_k^T + d2^k(S_kU_k^T)^T & -d2^kS_kJ_kU_k^T \\ d2^kS_kJ_kU_k^T & 2X_kU_k^T + d2^k(S_kU_k^T \end{array} \right] \quad (29)$$

where (29) follows from (28) by part 1 of Lemma 3 and by our inductive hypothesis. Then, to finish the proof, we must show that $S_mU_m^T$ is symmetric for all $m \geq 1$, and we do this sub-proof by induction on m.

The following base case for $k = 1$ is clearly symmetric:

$$\begin{aligned} S_1U_1^T &= \left[\begin{array}{cc} 1 & 1 \\ 1 & -1 \end{array} \right] \left[\begin{array}{cc} 1 & -1 \\ 1 & 1 \end{array} \right] \\ &= \left[\begin{array}{cc} 2 & 0 \\ 0 & -2 \end{array} \right] \\ &= (S_1U_1^T)^T \end{aligned}$$

Assume $S_kU_k^T = (S_kU_k^T)^T$ for some $k \geq 1$ an consider:

$$\begin{aligned} S_{k+1}U_{k+1}^T &= \left[\begin{array}{cc} S_k & S_k \\ S_kJ_k & -S_kJ_k \end{array} \right] \left[\begin{array}{cc} U_k^T & J_kU_k^T \\ U_k^T & -J_kU_k^T \end{array} \right] \\ &= \left[\begin{array}{cc} 2S_kU_k^T & 0 \\ 0 & 2S_kU_k^T \end{array} \right], \end{aligned}$$

which is symmetric by the inductive hypothesis. Thus, by induction, $S_mU_m^T$ is symmetric for all $m \geq 1$. We can now conclude that $X_{k+1}U_{k+1}^T = (X_{k+1}U_{k+1}^T)^T$, and thus by induction, $X_{k+1}U_{k+1}^T$ is also symmetric for all $m \geq 1$, completing our proof. □

Proof. (Part 3 by Induction)

The base case $k = 1$ is clearly symmetric:

$$\begin{aligned} S_1Y_1^T &= \left[\begin{array}{cc} 1 & 1 \\ 1 & -1 \end{array} \right] \left[\begin{array}{cc} b+e & -e \\ e & b+e \end{array} \right] \\ &= \left[\begin{array}{cc} 2b+e & e \\ e & -2b-e \end{array} \right]. \end{aligned}$$

Assume $S_kY_k^T = (S_kY_k^T)^T$ for some $k \geq 1$, and then consider:

$$S_{k+1}Y_{k+1}^T = \begin{bmatrix} S_{ky} & S_k \\ S_kJ_k & -S_kJ_k \end{bmatrix} \begin{bmatrix} Y_k^T + e2^kU_k^T & J_kY_k^T \\ Y_k^T & -J_kY_k^T - e2^kJ_kU_k^T \end{bmatrix}$$
$$= \begin{bmatrix} 2S_kY_k^T + e2^kS_kU_k^T & -e2^kS_kJ_kU_k^T \\ e2^kS_kJ_kU_k^T & 2S_kY_k^T + e2^kS_kU_k^T \end{bmatrix},$$

which can be seen to be symmetric because $S_kJ_kU_k^T$ is skew-symmetric by part 1 of Lemma 3 and $S_kY_k^T$ is symmetric by our inductive hypothesis. The result follows by induction. □

Proof. (Part 4 by Induction)

The base case of $k = 1$ holds as:

$$eX_1J_1U_1^T + dS_1J_1Y_1^T = e\begin{bmatrix} a & a+d \\ a+d & -a \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} +$$
$$d\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} b+e & -b \\ b & b+e \end{bmatrix}$$
$$= \begin{bmatrix} -de & -2ea-de \\ 2ea+de & -de \end{bmatrix} + \begin{bmatrix} de & -2db-de \\ 2db+de & de \end{bmatrix}$$
$$= \begin{bmatrix} 0 & -2ex-2dy-2de \\ 2ea+2db+2de & 0 \end{bmatrix}$$

Assume $e2^kX_kJ_kU_k^T + d2^kS_kJ_kY_k^T = -(e2^kX_kJ_kU_k^T + d2^kS_kJ_kY_k^T)^T$ for some $k \geq 1$, and consider $e2^kX_{k+1}J_{k+1}U_{k+1}^T + d2^kS_{k+1}J_{k+1}Y_{k+1}^T =$

$$\begin{bmatrix} -d2^ke2^kS_kU_k^T & 2e2^kX_kJ_kU_k^T + d2^ke2^kS_kJ_kU_k^T \\ 2e2^kX_kJ_kU_k^T + d2^ke2^kS_kJ_kU_k^T & d2^ke2^kS_kU_k^T \end{bmatrix} +$$
$$\begin{bmatrix} d2^ke2^kS_kU_k^T & 2d2^kS_kJ_kY_k^T + d2^ke2^kS_kJ_kU_k^T \\ 2d2^kS_kJ_kY_k^T + d2^ke2^kS_kJ_kU_k^T & -d2^ke2^kS_kU_k^T \end{bmatrix} =$$
$$\begin{bmatrix} 0 & 2(e2^kX_kJ_kU_k^T + d2^kS_kJ_kY_k^T + d2^ke2^kS_kJ_kU_k^T) \\ 2(e2^kX_kJ_kU_k^T + d2^kS_kJ_kY_k^T + d2^ke2^kS_kJ_kU_k^T) & 0 \end{bmatrix},$$

which is skew-symmetric by part 1 of Lemma 3 and by our inductive hypothesis. Therefore, the desired result holds by induction. □

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